The Effects of Correlated Noise
in Phased-Array Observations of Radio Sources

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Abstract

Arrays of radio telescopes are now routinely used to provide increased signal-to-noise when observing faint point sources. However, calculation of the achievable sensitivity is complicated if there are sources in the field of view other than the target source. These additional sources not only increase the system temperatures of the individual antennas, but may also contribute significant "correlated noise" to the effective system temperature of the array. This problem has been of particular interest in the context of tracking spacecraft in the vicinity of radio-bright planets (*e.g.* Galileo at Jupiter), but it has broader astronomical relevance as well. This paper presents a general formulation of the problem, for the case of a point-like target sourer in the presence of an additional radio source of arbitrary brightness distribution. We re-derive the well known result that, in the absence of any background sources, a phased array of N identical antennas is a factor of N more sensitive than a single antenna. We also show that an *unphased* array of N identical antennas is, on average, no more sensitive tha n a single antenna if the signals from the individual antennas are combined prior to detection. In the case where a background source is present we show that the effects of correlated noise are highly geometry dependent, and for some astronomical observations may cause significant fluctuations in the array's effective system temperature.

1 Introduction

Arrays of radio telescopes can be used not only to map extended radio sources, and make high accuracy astrometric measurement of point sources, but, in a "phased-array" mode, to provide increased signal-to-noise when observing faint point sources. For example, the phased-array VLA is now regularly used for pulsar timing observations ($e.g.$ Thorsett 1991) and VLBI experiments (Thompson, Moran and Swenson 1986, and references therein), and was arrayed with antennas at the the Deep Space Network complex in Gold dstone for increased sensitivity in telemetry reception during Voyager's Neptune encounter (Stone and Miner 19S9; 1 Brown $et~al.$ 1990); a variety of arraying schemes are being; considered for telemetry reception during the Galileo spacecraft's tour of Jupiter (G. Resch, personal communication). Though there have been extensive discussions of the sensitivity that can be obtained in an interferometric image ($e.g.$ Thompson, Moran and Swenson 1986, [pp].] 55-1 68]; Perley, Schwab and Bridle 1 989 [Chaps.7,22,23]; McCullough 1 993), treatments of non-imaging applications of arrays have been limited ($e.g.$ Thompson, Moran and Swenson 1986 [pp.308-31 O]), and in some cases, not entirely accurate. This paper presents a general formulation of the effective gain and the effective system temperature of an array of antennas, and Id discusses in depth one of the problems peculiar to the use of phased arrays to obtain large collecting areas, that of the "correlated noise" from extraneous sources in the array's field of view. This problem has been of particular interest in the context of tracking spacecraft in the vicinity of radio-bright planets ($e.g.$ Galileo at Jupiter [Dewey]()()2]), but it has broader astronomical relevance as well, particularly to the calibration of VLBI experiments.

The paper is organized as follows: Section 2 derives expressions for the effective gain and system temperature of an array, and for the contribution of correlated noise to the effective system temperatures. The case considered in fairly general: a lmilltl-like target source in the presence of an additional radio source of arbitrary brightness distribution, observed by

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a array of N not-necessarily-identical telescopes. In Section 3 we use these expressions to examine the effects of correlated noise for a realistic astronomical observing situation, which illustrates that the effects of correlated noise may be large (comparable to or greater than other contributions to the effective system temperature) and may vary significantly on short time scales. For the sake of compactness the derivations of a number of expressions used in main text are relagated to appendices. Appendix A examines the voltage fluctuations in the arrayed signal, Appendix B derives expressions for the spectral power in the arrayed signal, and Appendix C derives the optimal form of the weighting factors to use when combining the signals from the individual antennas.

2 Effective gain and system **temperature** of an array

For a single antenna of gain G and system temperature T observing a source of flux S_0, the signal-to-noise in a single polarization and unit time and frequency interval can be written

$$\mathcal{R}_{SN} = \frac{G}{T} S_0 \quad . \tag{1}$$

Note that all the information about the sensitivity of the antenna and the front-end receiver is contained in the term G/T'; S_0 depends only on the source. The quantity G/T is commonly used as a "figure-of-merit" to describe the sensitivity of a particular antenna.

Here we derive an effective gain, G_{eff} and an effective system temperature T_{eff} for an array. Analogous to the single antenna case, these quantities must satisfy

$$\mathcal{R}_{SN} = \frac{G_{eff}}{T_{eff}} S_0 \quad . \tag{2}$$

If the total received power (including system noise) per unit bandwidth in a single polarization, is P_Σ, and P_Σ^{target} is the portion due to the target source, the signal-to-noise

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can be written

$$\mathcal{R}_{SN} = \frac{P_{\Sigma}^{target}}{\Delta P_{\Sigma}} \tag{3}$$

where ΔP_{Σ} represents the root-mean-square fluctuations in the arrayed spectral power. It is shown in Appendix A that when the signals from the array elements are summed before detection $\Delta P_{\Sigma} = P_{\Sigma}$ just as for a single antenna. Equation (3) can then be written so

$$\sim . \ s \ e \ ! = \frac{P_{\Sigma}^{target}}{} \ j \ i \ - \ ; \tag{4}$$

effective system temperature of the array as can be identified as

$$T_{eff} = \frac{P_{\Sigma}}{k_B} \tag{5}$$

and the effective gain of the array as

$$G_{eff} = \frac{P_{\Sigma}^{target}}{k_B S_0} \ . \tag{6}$$

In this paper we consider the case where the "target source" is point-like, has a total flux S_0 and is located in a direction $\hat{s}_0$. For the purposes of this discussion "point-like" can be taken to mean small compared to the synthesized beam of the array or, equivalently, unresolved on any array baseline. We assume that there is also a background source (or sources) in the array's field of view, with total flux S^{bkgnd} and brightness distribution $I^{bkgnd}(\hat{s})$. For simplicity we assume that both S_0 and $I^{bkgnd}(\hat{s})$ are constant in time and do not vary appreciably with frequency across the bandwidth used for the observations.

We assume that the array is composed of N antennas, not necessarily identical. The coordinates of the ith antenna are represented by the vector r_i, its on-axis gain by G_i and its system temperature (looking at an "empty" field) by T_i. In addition we assume that all the antennas in the array are pointed in the direction $\hat{s}_0$ (i. e. toward the target source) and that all the antennas use the same bandwidth.

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The purpose of arraying is to combine the signals from the N antennas in such a way as to maximize the sig]]al-tfo-noise on the target source. To this end, undetected signals from each antenna are delayed, phase-shifted, and summed, with appropriately chosen weighting factors (Thompson, Moran and Swenson 1986 [pp.308-310]). If $v_i(\nu, t)$ represents voltage J-rom i^{th} antenna at time t and frequency v (see Appendix B, the arrayed signal can be written

$$v_\Sigma(\nu, t) = \sum_{i=1}^{N} W_i e^{-i\phi_i^m} v_i(\nu, t + \tau_i^m) \ . \tag{7}$$

The inserted "model" delays, τ_i^m, are chosen so as to compensate for delays suffered by a signal from direction $\hat{s}_0$, and can be calculated *a priori*. When the array is properly phased, the values of ϕ_i^m, the inserted "model" phase-shiftjs, are chosen so as to compensate for random phase differences between antenna, caused by media and hardware effects. These phase corrections are not known o *priori* and must be determined in real time. The weighting factors, W_i, can be chosen to maximize: signal-to-noise in the summed signal. We assume they are normalized such that $\sum_i W_i^2 = 1$, and when called for, use the approximation $W_i \propto \sqrt{G_i/T_i}$ which is strictly valid when the noise contribution Of the background source can be ignored (see Appendix C). It should be noted that the details of how the delays and phases are inserted, or the frequency to which the signal is mixed before various stages of the processing are are irrelevant for the purpose of this calculation.

The total power be unit bandwidth in the summed signal is given by

$$
\begin{aligned}
P_\Sigma &= \left\langle |v_\Sigma(\nu, t)|^2 \right\rangle_T \\
&= \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \left\langle e^{i(\phi_k^m - \phi_i^m)} v_i(\nu, t + \tau_i^m)[v_k(\nu, t + \tau_k^m)]^* \right\rangle_T \ ,
\end{aligned}
\tag{8}
$$

where angle brackets denote time averaging, and * denotes complex conjugate.

As shown in Appendix B this can be re-written in a more physically informative form,

$$P_\Sigma = k_B \left[\sum_{i=1}^{N} W_i^2 T_i + \iint d\hat{s}\, B(\hat{s} - \hat{s}_0) I(\hat{s}) \right] \ , \tag{9}$$

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where $B(\hat{s} - \hat{s}_0)$ represents the synthesized beam of the array (see Equation [46]), and $I(\hat{s})$ is the total brightness distribution, including target source. In general the synthesized beam $B(\hat{s} - \hat{s}_0)$ has a large number of high sidelobes; therefore regions of the source well removed from $\hat{s}_0$ may contribute significantly to the summed power. The more compact the array is, the larger the region of sky covered by the synthesized beam pattern. If the brightness distribution I^{bkgnd} is patchy, the amount of power contributed by the sidelobes will change as the orientation of the synthesized beam pattern on the sky changes.

Appendix B also shows that P_Σ can be written in terms of the source visibilities,

$$P_\Sigma = k_B \left[\sum_{i=1}^{N} W_i^2 T_i + S_0 \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} + \right.$$
$$\left. + S^{bkgnd} \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} V_{ik}^{bkgnd} \right] \quad , \qquad (10)$$

where V_{ik}^{bkgnd} is the *normalized* visibility of the background source on the i-k baseline (see Equation [48]), S^{bkgnd} is the total flux of the background source, and $\delta\phi_i = \phi_i - \phi_i^m$ is the difference between the actual phaseshift, ϕ_i at the ith antenna and the inserted phase correction, ϕ_i^m.

The summed power due only to the target source, P_Σ^{target} is

$$P_\Sigma^{target} = k_B S_0 \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \quad , \qquad (11)$$

so from Equation (6) it follows that

$$G_{eff} = \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} = \sum_{i=1}^{N} W_i^2 G_i + \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \quad . \qquad (12)$$

If we assume that the contribution of the target source to the total power is small, *i.e.*

$$S_0 \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k} \ll \sum_{i=1}^{N} W_i^2 T_i + S^{bkgnd} \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k} \quad , \qquad (13)$$

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so it follows from Equations (5) and (1O) that

$$T_{eff} = \sum_{i=1}^{N} W_i^2 T_i + S^{bkgnd} \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \mathcal{V}_{ik}^{bkgnd}$$

$$= \sum_{i=1}^{N} W_i^2 \left[T_i + S^{bkgnd} \bar{G}_i \right] + S^{bkgnd} \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \mathcal{V}_{ik}^{bkgnd} \quad ,$$

(14)

where $\bar{G}_i$ represents the gain of the ith antenna in the direction of the background source. The first term in the above equation is the weighted average of the system temperatures (including the contribution from the background source) of the N antennas. The second term in Equation (14), the sum over all baselines, represents the correlated noise,

$$T_{corr} = S^{bkgnd} \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \mathcal{V}_{ik}^{bkgnd} \quad .$$

(15)

Both G_{eff} and T_{eff} depend, through $\delta\phi_i - \delta\phi_k$, on the phase shifts ϕ_i^m, and ϕ_k^m which are inserted to correct for the random antem]]a-based phase shifts. The effective gain, and in all but pathological cases, the signs]-to-noise, will be maximized when $\delta\phi_i - \delta\phi_k = 0$ for all i, k. If this condition cannot be fulfilled, G_{eff} will be less than its optimum value. Fortunately, since the quantities $\mathcal{V}_{ik}$ are complex, phasing the array dots not necessarily maximize T_{eff}. For a perfectly phased array ($\delta\phi_i = \delta\phi_k = 0$) the effective gain and system temperature reduce to

$$G_{eff}^{\phi} = \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}$$

(16)

a 1") d

$$T_{eff}^{\phi} = \sum_{i=1}^{N} W_i^2 \left[T_i + S^{bkgnd} \bar{G}_i \right] + S^{bkgnd} \sum_{i=1}^{N} \sum_{k \neq i}^{N} W_i W_k \sqrt{G_i G_k}\, \mathcal{V}_{ik}^{bkgnd} \quad .$$

(17)

These expressions take particularly simple forms in the case of an array of identical antennas, where for all i, $W_i = 1/\sqrt{N}$, $G_i = G$ and $T_i = T$:

$$G_{eff}^{\phi} = NG \quad ,$$

(18)

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$$T_{\text{eff}}^{\phi} = T + \dot{G}S^{bkgnd} + GS^{bkgnd}\frac{1}{N}\sum_{i=1}^{N}\sum_{k\neq i}^{N}\mathcal{V}_{ik}^{bkgnd} \quad , \tag{19}$$

and

$$T_{corr} = GS^{bkgnd}\frac{1}{N}\sum_{i=1}^{N}\sum_{k\neq i}^{N}\mathcal{V}_{ik}^{bkgnd} \quad . \tag{20}$$

Though for an extended array T_{corr} approaches zero, for a compact array it may become the dominant contribution to T_{eff}^{ϕ}. For a compact array of identical antennas $T_{\text{eff}}^{\phi} \sim (A^{\tau}-1)\dot{G}S^{bkgnd}$. It should be noted that for some geometries $T_{corr} < 0$; however the total contribution of the background source, $(\dot{G}S^{bkgnd} + T_{corr}$, for identical antennas) is always positive.

It is also instinctive to consider the case of an unphased array of identical antennas. In this case $\phi_i^m = \phi_k^m = 0$ and $\delta\phi_i = \phi_i$, $\delta\phi_k = \phi_k$. Since the antenna based phase-shifts ϕ_i, ϕ_k are uncorrelated between antennas and vary with time, the effective gain and system temperature vary with time as WC]]. For truly random phases, the sum over the $N(N-1)$ antenna pairs in Equations (12) and (14) is, on average, zero. It follows that, for an unphased array, the average values of G_{eff} and T_{eff}, T_{corr} are given by

$$\langle G_{\text{eff}}^{un-\phi}\rangle = G \quad , \tag{21}$$

$$\langle T_{\text{eff}}^{un-\phi}\rangle = T + S^{bkgnd}\bar{G} \quad , \tag{22}$$

and

$$\langle T_{corr}\rangle = 0 \quad . \tag{23}$$

It has **been** stated (Thompson, Moran ant] Swenson 1986 [p.309]) that an unphased array is more sensitive, by a factor of $\sqrt{N}$, than an single element. However, it is clear from the above analysis, that this is not the case. The response of an unphased array of identical antenna is, on average, identical to the response of a single element, if the antenna phases (ϕ_i, ϕ_k) are truly ujlcol'l'elated. This is a direct result of the fact that when the signals from

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the individual antenna.s are summed prior to detection, the power in the summed sigma] due to the target source grows no faster than the power due to noise, unless the array is phased (scc Appendixscc-noise). A $\sqrt{N}$ increase in sig]]al-to-noise can be obtained from an unphased array if the signals from the individual antennas are detected before the summing is done. In this case, there is no further improvement if the array is phased. An analogy can be made to the case where signals of opposite polarizations are summed. If the signals are added after detection sensitivity is increased by a factor of $\sqrt{2}$; however if signal are combined before detection as many components cancel as add in phase,

3 Discussion

It is useful to examine the effects of correlated noise in the context of a realistic astronomical observing scenario. We consider the case where the phased-array VLA is used as an element in C-band VLBI observations of the core of a radio galaxy similar to Cygnus A

The VLA is modelled as an array of 27 identical 25 meter antennas. We use the C-band (5 G]] z) antenna parameters listed in the (*VLA Observational Status Summary, May 1993*), specifically, 0.1 K/Jy gain, (; 0 1< zenith system temperature, and 9 arcminute half-power beam. We use the antenna spacings listed in Thompson *et* al. (19S0); minimum and maximum baselines are, respectively, 0.S and 36 km (A-configuration), 0.2 and 11 km (B-configuration), 0.0S and 3.4km (C-configuration), and 0.04 and 1.0 km (D-configuration).

The qualitative effects of correlated noise can been seen with a quite crude model of a radio galaxy with a faint core, and bright lobes. We a use model patterned roughly after Cygnus A (Mitten and Ryle 1969; Carilli cl *al.* 1991): a radio galaxy with lobes described by two circular gaussians (40 arcsecond 1/c width), separated by 2 arcminutes with the core midway between them. There are hot spots at the edge of each lobe also described by circular gaussians (4 arcsecond 1/c width). The total C-band flux (excluding the core) is assumed to

be 350 Jy, of which 300 Jy is in the lobes, and 50 Jy in the hot spots. In order to assess the effects of different source geometry on the correlated noise we calculate T^{ϕ}_{eff} for 45 different orientations of the source on the sky. Other changes in geometry (such as varying the exact location of the core or the hot spots) have a much smaller effect on T^{ϕ}_{eff} than changes in the overall source orientation. Figure 1 plots T^{ϕ}_{eff} versus hour angle for the four different array configurations, and 45 different orientations of the surce on the sky. Each point of the graph corresponds to the effective temperature at a given hour angle, for a particular, randomly chosen, source orientation. The variation of T_{eff} with hour angle for one source orientation (that corresponding approximately the true orientation of Cyg A) is shown as a solid line to indicate the timescale on which T^{ϕ}_{eff} varies. It is clear from the large scatter of points in Figure 1 that T_{eff} is highly geometry dependent, and therefore may vary significantly with hour angle. The source contributes about 35 K to the system temperature of each antenna, and for all but the most compact configuration the *average* contribution of correlated noise is zero. In the compact D-configuration the average correlated noise contribution is 8 K, but it is highl y variable: over all the geometries considered, the minimum 1)-array correlated noise contribution is – 101< and the maximum is -| 1351<. The variability is less pronounced in the more extended array configurations: in C-array the minimum correlated noise contribution is - 8 K, the maximum +-25 K; in B-array the minimum is –4 K, the maximum +6; in A-array the minimum is –1 K and the maximum, +2 K.

For sources with less small-scale structure the correlated noise contribution will show less variability on short time scales, but may still be significant,. For the Crab supernova remnant, with a diameter $\sim$ 3 arcminutes and an L-band flux of $\sim$ 1000 Jy, the correlated noise contribution in L-band phased-VLA (1 l-array) observations ranges smoothly from $\sim$ 100 K at transit, to $\sim$ 2501< at large hour angles; this contribution is larger than the $\sim$ 90 K contribution of the remnant to the system temperature of an individual antenna.

With current receiver sensitivities there only a few sources where the correlated noise

effects will be as pronounced as in the above examples. However there are many sources for which correlated noise may have a measurable effect on sensitivity which should be accounted for when designing experiments and calibrating data.

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Figure Caption

Figure 1: This figure plots T_{eff}^{ϕ} vs. hour angle for the case where the phased-array VLA is used to observe the core of a radio galaxy similar to Cygnus A; the four panels correspond to the four array configurations of the VLA (A, B, C, D). Each panel shows T_{eff}^{ϕ} vs. hour angle points for 45 orientations on the sky; each point plotted corresponds to T_{eff}^{ϕ} for a given hour angle and given source orientation. For one source orientation (approximately that of Cyg A) T_{eff}^{ϕ} VS. hour angle is shown as a solid line in order to indicate the timesc.ales on which T_{eff}^{ϕ} vanes.

Appendices

A Fluctuations in the arrayed signal

In this section we consider the fluctuations in spectral power of the summed signal, and motivate the statement that $\Delta P_\Sigma = P_\Sigma$, where P_Σ is the spectral power of the arrayed signal. The completely generally derivation is left to readers with a large supply of scratch paper. The basic expression for the fluctuations in the instantaneous spectral power is

$$\Delta P_\Sigma = \langle (|v_\Sigma(\nu,t)|^2 - P_\Sigma)^2 \rangle^{1/2} = \left[\langle |v_\Sigma(\nu,t)|^4 \rangle - P_\Sigma^2 \right]^{1/2} , \tag{24}$$

where v_Σ is given by Equation (7). For this calculation it is useful to work with the real and imaginary parts of the signals explicitly and we will write

$$e^{-i\phi_i^m} v_i(v, t + \tau_i^m) = x_i + iy_i , \tag{25}$$

(such that $v_\Sigma = \sum_i^N [x_i + y_i)]$), where x_i, y_i are zero-mean random variables. We assume that for a given antenna x_i and y_i are uncorrelated, and that they have identical statistical properties (mean, variance etc.). It is useful to note that with these assumptions

$$\langle x_i \rangle = \langle y_i \rangle = 0 \qquad \text{and} \qquad \langle x_i^2 \rangle = \langle y_i^2 \rangle = \frac{P_i}{2} , \tag{26}$$

where P_i is the spectral power at the ith antenna. Using the fact that for a single antenna, $\Delta P_i = P_i$, we can write

$$P_i^2 = \langle [2/ + y_i^2]^2 \rangle - P_i^2 = \langle x_i^4 \rangle + \langle y_i^4 \rangle + 2\langle x_i^2 \rangle \langle y_i^2 \rangle - P_i^2 \tag{27}$$

from which it follows that

$$\langle x_i^4 \rangle = \langle y_i^4 \rangle = \frac{3P_i^2}{4} . \tag{2s}$$

Returning to the summed signal, the spectral power in the summed signal can be written

$$|v_\Sigma(\nu, t)|^2 = \sum_{i=1}^N W_i(x_i + iy_i) \sum_{k=1}^N W_k(x_k - iy_k)$$

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$$= \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k x_i x_k + \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k y_i y_k \quad , \tag{29}$$

so,

$$P_\Sigma = \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k (\langle x_i x_k \rangle + \langle y_i y_k \rangle) \quad . \tag{30}$$

and

$$|v_\Sigma(\nu,t)|^4 = \left[\sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k x_i x_k + \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k y_i y_k\right]^2$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} W_i W_k W_l W_m x_i x_k x_l x_m + \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} W_i W_k W_l W_m y_i y_k y_l y_m +$$

$$+2\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} W_i W_k W_l W_m x_i x_k y_l y_m \quad . \tag{31}$$

In the interests of space we will not try work with these expressions in their completely general form. Rather we will consider the case of an array of identical antennas ($W_i = W_k = W$, $P_i = P_k = P$) where Equation (31) becomes

$$|v_\Sigma(\nu,t)|^4 = W^4 \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_i x_k x_l x_m + W^4 \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} y_i y_k y_l y_m +$$

$$+2W^4 \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_i x_k y_l y_m \tag{32}$$

and Equation (30) becomes

$$\bar{P}_\Sigma = W^2 \sum_{i=1}^{N}\sum_{k=1}^{N} (\langle x_i x_k \rangle + \langle y_i y_k \rangle) \quad , \tag{33}$$

We then examine two limiting cases and show that $AP_E = P_\Sigma$ holds for both. In the first case we assume that the signals at the antennas are completely uncorrelated; in the second we will assume that the signal at all the antennas are completely correlated, in fact that signals at one antenna is exactly the same as the signal at every other antenna.

In the case where the signals at all antennas are uncorrelated we note that for $i \neq k$

$$\langle x_i x_k \rangle = \langle x_i \rangle \langle x_k \rangle = 0 \qquad and \qquad \langle x_i x_k^3 \rangle = \langle x_i \rangle \langle x_k^3 \rangle = 0 \quad . \tag{34}$$

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Using these expressions, Equation (33) becomes

$$P_\Sigma^{uncorr} = W^2 \sum_{i=1}^{N} \left(\langle x_i^2 \rangle + \langle y_i^2 \rangle \right) = W^2 N P \quad . \tag{35}$$

and Equation (32) becomes

$$\begin{aligned}
\left\langle |v_\Sigma^{uncorr}|^4 \right\rangle &= W^4 \left\langle \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_i x_k x_l x_m \right\rangle + W^4 \left\langle \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} y_i y_k y_l y_m \right\rangle + \\
&\quad + 2W^4 \left\langle \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_i x_k y_l y_m \right\rangle \\
&= W^4 \left\langle \sum_{i=1}^{N} x_i^4 + 3\sum_{i=1}^{N}\sum_{k\neq i}^{N} x_i^2 x_k^2 \right\rangle + W^4 \left\langle \sum_{i=1}^{N} y_i^4 + 3\sum_{i=1}^{N}\sum_{k\neq i}^{N} y_i^2 y_k^2 \right\rangle + \\
&\quad + 2W^4 \left\langle \sum_{i=1}^{N}\sum_{l=1}^{N} x_i^2 y_l^2 \right\rangle \\
&= W^4 \sum_{i=1}^{N} \langle x_i^4 \rangle + 3W^4 \sum_{i=1}^{N}\sum_{k\neq i}^{N} \langle x_i^2 \rangle \langle x_k^2 \rangle + W^4 \sum_{i=1}^{N} \langle y_i^4 \rangle + \\
&\quad + 3W^4 \sum_{i=1}^{N}\sum_{k\neq i}^{N} \langle y_i^2 \rangle \langle y_k^2 \rangle + 2W^4 \sum_{i=1}^{N}\sum_{l=1}^{N} \langle x_i^2 \rangle \langle y_l^2 \rangle \\
&= 2W^4 \left[3N\frac{P^2}{4} + 3N(N-1)\frac{P^2}{4} + N^2 \frac{P^2}{4} \right] \\
&= 2W^4 N^2 P^2 \quad . \tag{36}
\end{aligned}$$

When these are expressions are substituted into Equation (24) we have

$$\Delta P_\Sigma^{uncorr} = \left[\langle |v_\Sigma^{uncorr}|^4 \rangle - P_\Sigma^{uncorr\,2} \right]^{1/2} = W^2 N P = P_\Sigma^{uncorr} \tag{37}$$

as required.

For the completely correlated case we can substitute $x_i = x_k = x$ and $y_i = y_k = y$ so

$$\begin{aligned}
\left\langle |v_\Sigma^{corr}|^4 \right\rangle &= \left\langle W^4 N^4 (x^4 + y^4 + 2x^2 y^2) \right\rangle \\
&= W^4 N^4 (\langle x^4 \rangle + \langle y^4 \rangle + 2\langle x^2 \rangle \langle y^2 \rangle) \\
&= 2W^4 N^4 P^2 \tag{38}
\end{aligned}$$

and

$$P_\Sigma^{corr} = W^2 N^2 (\langle x^2 \rangle + \langle y^2 \rangle) = W^2 N^2 P \quad . \tag{39}$$

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Equation (24) then reduces to

$$\Delta P_\Sigma^{corr} \cdot W^2 N^2 P = P_\Sigma^{corr} \tag{40}$$

as required.

B Expression for single-dish, correlated, and arrayed spectral power

The **voltage at the** i^{th} antenna can be written

$$v_i(\nu,t) = e^{i\phi_i}\sqrt{k_B}\iint d\hat{s}\, g_i(\hat{s}-\hat{s}_0)\mathcal{E}_0(\nu,\hat{s})e^{i2\pi\nu[t-\frac{\mathbf{r}_i\cdot\hat{s}}{c}]} + v_i^n(\nu,t) \ . \tag{41}$$

The first term in this equation, v_i^n, is the random noise voltage, and the second term is the voltage clue to celestial sources, where $\mathcal{E}_0(\nu,\hat{s})$ is the electric field at the origin of the array, clue to radiation from the direction $\hat{s}$ at frequency ν; $g_i(\hat{s}-\hat{s}_0)$ is the field pattern of the ith antenna (defined such that $g_i^2(0)=G_i$, where G_i is the on-axis gain of the ith antenna); $\mathbf{r}_i$ is the displacement i^{th} of the antenna from the origin; ϕ_i is a phaseshift clue to media and instrumental effects particular to the ith antenna; c is the speed of light and k_B is Boltzmann's constant.

The correlated spectral power can be written

$$\begin{aligned}
\rho_{ik}(\nu) &= \left\langle e^{i(\phi_k^m-\phi_i^m)}v_i(\nu,t+\tau_i^m)[v_k(\nu,t+\tau_k^m)]^*\right\rangle_T \\
&= k_B\left\langle e^{i(\phi_k^m-\phi_i^m)}e^{i\phi_i}\left[\iint d\hat{s}\,g_i(\hat{s}-\hat{s}_0)\mathcal{E}_0(\nu,\hat{s})e^{i2\pi\nu[t+\tau_i^m-\frac{\mathbf{r}_i\cdot\hat{s}}{c}]}+v_i^n(\nu,t+\tau_i^m)\right]^x \right. \\
&\qquad \left. \left[e^{i\phi_k}\iint d\hat{s}\,g_k(\hat{s}-\hat{s}_0)\mathcal{E}_0(\nu,\hat{s})e^{i2\pi\nu[t+\tau_k^m-\frac{\mathbf{r}_i\cdot\hat{s}}{c}]}+v_k^n(\nu,t+\tau_k^{\overset{*}{m}})\right]\right\rangle_T \ .
\end{aligned} \tag{42}$$

Since radiation from natural radio sources is spatially incoherent, and the noise voltage at antenna i is uncorrelated with that at antenna k, for $i\neq k$ this reduces to

$$\rho_{ik}(\nu) = k_B e^{i(\delta\phi_i-\delta\phi_k)}\iint d\hat{s}\,g_i(\Delta s)g_k(\Delta s)I(\hat{s})e^{i2\pi\nu\mathbf{B}_{ik}\cdot\Delta s} \quad, \qquad i\neq k \tag{43}$$

where $\Delta s = \hat{s} - \hat{s}_0$ and $\mathbf{B}_{ik} = (\mathbf{r}_k - \mathbf{r}_i)/c$, $\delta\phi_i = \phi_i - \phi^m$, and $I(\hat{s})$ is the combined brightness distribution of all sources in the beam. For $i = k$ this corresponds to P_i the single dish power of the ith antenna,

$$\rho_{ii}(\nu) = P_i(\nu) = k_B \iint d\hat{s}\, g_i^2(\Delta s) I(\hat{s}) + k_B T_i \quad . \tag{44}$$

in the above equations $\delta\phi_i = \phi_i - \phi^m$, $I(\hat{s})$ is the combined brightness distribution of all sources in the beam, T_i is the noise temperature of the ith antenna ant] k_B is Boltzmann's constant.

The total summed power can then be written

$$P_\Sigma = k_B \sum_{i=1}^{N} W_i^2 T_i + k_B \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k e^{i(\delta\phi_i - \delta\phi_k)} \iint d\hat{s}\, g_i(\Delta s) g_k(\Delta s) I(\hat{s}) e^{i2\pi\nu \mathbf{B}_{ik}\cdot\Delta s} \quad . \tag{45}$$

If we note hat the synthesised beam of the array is given by

$$B(\Delta s) = \sum_{i=1}^{N} \sum_{k=1}^{N} W_i W_k e^{i(\delta\phi_i - \delta\phi_k)} g_i(\Delta s) g_k(\Delta s) e^{i2\pi\nu \mathbf{B}_{ik}\cdot\Delta s} \quad , \tag{46}$$

it can be seen that

$$P_\Sigma(\nu) = k_B \left[\sum_{i=1}^{N} W_i^2 T_i + \iint d\hat{s}\, B(\Delta s) I(\hat{s}) \right] \quad . \tag{47}$$

The contribution of celestial sources to the summed power is given by the integral over the sky of the product of the synthesized beam and the brightness distribution. Since for most arrays the synthesized beam has significant side-lolxw, regions of the source well away from $\hat{s}_0$ may make a significant contribution to P_Σ.

Alternatively, if we note that the *normalized* source visibility $\mathcal{V}_{ik}$ on the i-k baseline is given by

$$\mathcal{V}_{ik} = \frac{1}{S\sqrt{G_i G_k}} \int\int d\hat{s}\, g_i(\Delta s) g_k(\Delta s) I(\hat{s}) e^{i2\pi\nu \mathbf{B}_{ik}\cdot\Delta s} \quad , \tag{4s}$$

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where $S = \iint d\hat{s} I(\hat{s})$. For the special case of $i = k$,

$$\mathcal{V}_{ii} = \frac{1}{G_i S} \iint d\hat{s}\, g_i^2(\Delta s) I(\hat{s}) = \frac{\dot{G}_i}{G_i} \sim 1 \quad , \tag{49}$$

where G_i is the weighted average, over the brightness distribution, of' the gain of the ith antenna. The summed power can be written

$$P_\Sigma(\nu) = k_B \left[\sum_{i=1}^{N} W_i^2 T_i + S \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \mathcal{V}_{ik} \right] \quad . \tag{50}$$

Separating the contributions of the target source and the background source gives

$$P_\Sigma(\nu) = k_B \left[\sum_{i=1}^{N} W_i^2 T_i + S_0 \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} + \right.$$
$$\left. + S^{bkgnd} \sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k}\, e^{i(\delta\phi_i - \delta\phi_k)} \mathcal{V}_{ik}^{bkgnd} \right] \quad . \tag{51}$$

C Weighting factors

The weighting factors W_i should be chosen so as to maximize the ratio $G_{eff}^\phi / T_{eff}^\phi$, where G_{eff}^ϕ and T_{eff}^ϕ are given by Equations (1 6) and (1 7), respectively. Such W_i will satisfy the condition

$$\frac{\partial}{\partial W_j}\left(\frac{G_{eff}^\phi}{T_{eff}^\phi} \right) = \frac{1}{{T_{eff}^\phi}^2}\left[T_{eff}^\phi \frac{\partial G_{eff}^\phi}{\partial W_j} - G_{eff}^\phi \frac{\partial T_{eff}^\phi}{\partial W_j} \right] = 0 \quad , \tag{52}$$

for all $i = 1, ... N$, where $G_{eff}^\phi, T_{eff}^\phi$ are given by Equations (16) and (17). in the case where the contribution of S^{bkgnd} can be ignored, these conditions can be re-written as

$$0 = \left[\sum_{i=1}^{N} W_i^2 T_i \right]\left[2W_j G_j + 2\sqrt{G_j}\sum_{i \neq j} W_i \sqrt{G_i} \right] - [2W_j T_j]\left[\sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k} \right]$$
$$= W_j G_j \left[\sum_{i=1}^{N} W_i^2 T_i \right] + \left[\sqrt{G_j}\sum_{i \neq j} W_i \sqrt{G_i} \right]\left[\sum_{i=1}^{N} W_i^2 T_i \right] - W_j T_j \left[\sum_{i=1}^{N}\sum_{k=1}^{N} W_i W_k \sqrt{G_i G_k} \right]$$
$$\tag{53}$$

for all $j = 1, ... N$. These N equations determine the N values of W_j to within an overall normalization, and it is not hard to verify they are satisfied by

$$W_j \propto \frac{\sqrt{G_j}}{T_i} \quad , \qquad j = 1...N \quad . \tag{54}$$

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It is usually convenient to choose the normalization such that

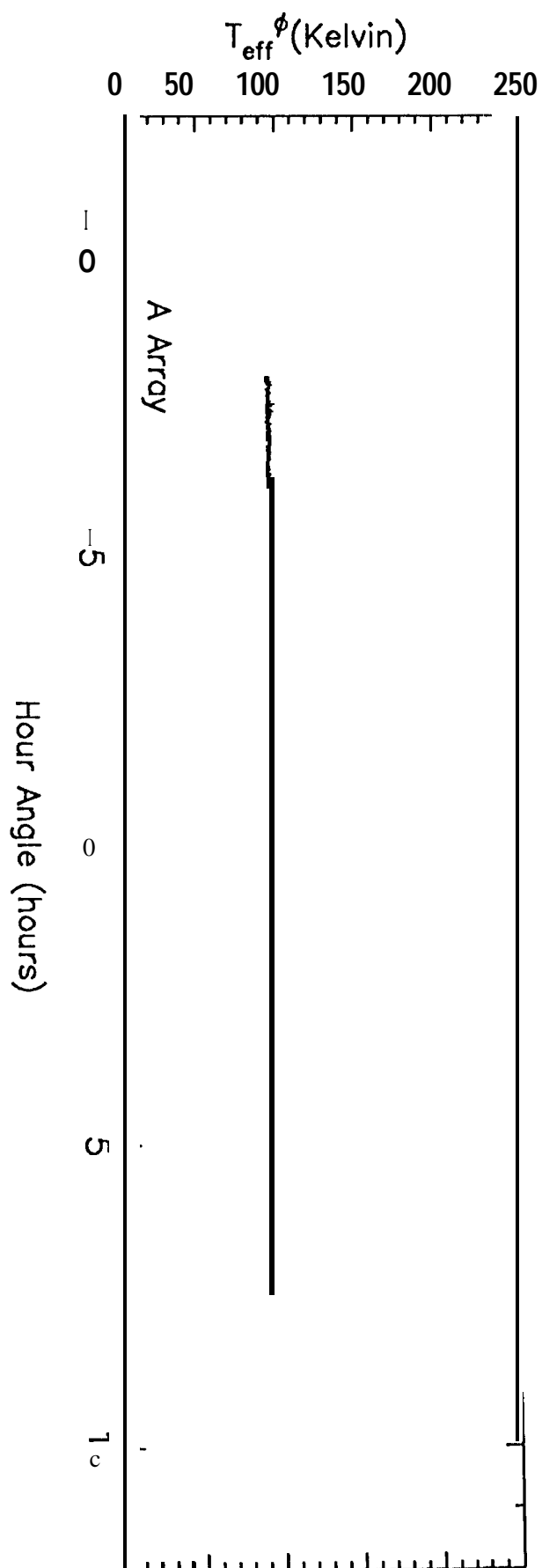$$\sum_{i=1}^{N} W_i^2 = 1 \quad . \tag{55}$$

Note that the optimal weighting factors, W_j, are not proportional to the signal-to-noise ratios, R_{SNj} (see Equation [[]]).

If the contribution of any background sources the system temperature of at each antenna is accounted for, but the correlated noise terms are ignored, the optimal weighting is
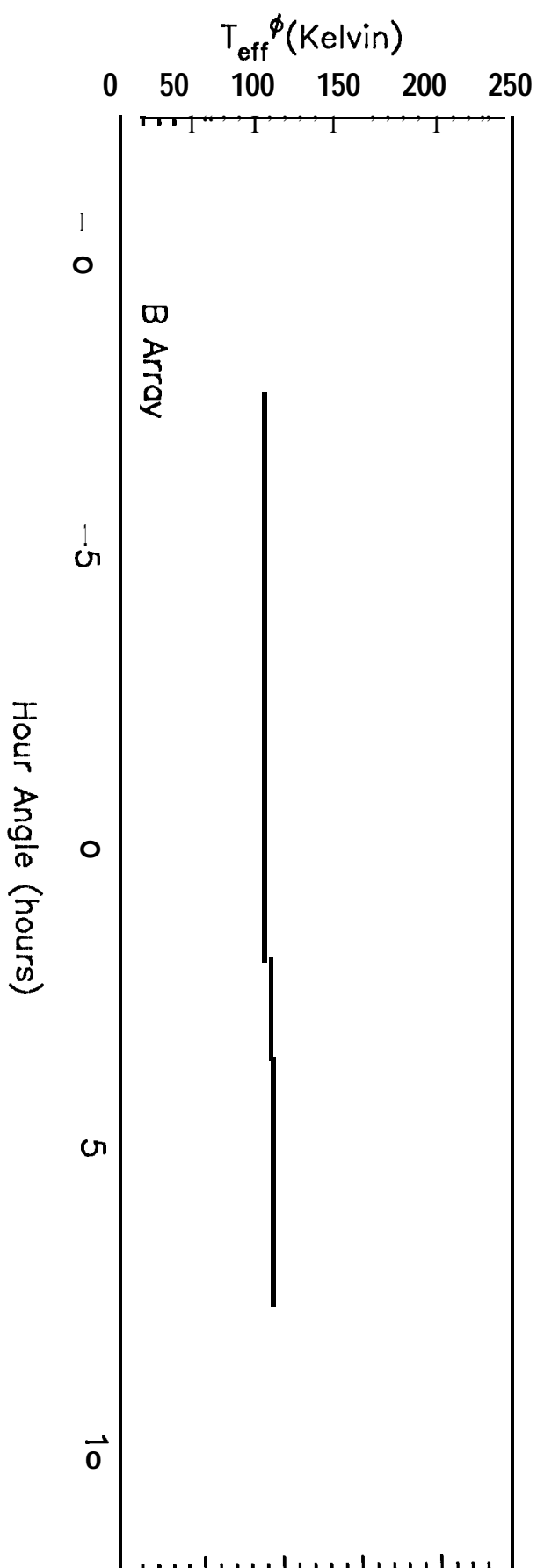
$$W_j \propto \frac{\sqrt{G_i}}{T_i + G_i S^{s\ bkgnd}} \quad , \qquad j = 1...N \quad . \tag{56}$$

If correlated noise terms are included there is no simple closed-form expression for the weighting factors, but it is straight forward to solve for them numerically if necessary. For most purposes loss of signal-to-noise caused by using Equation (56) is insignificant.
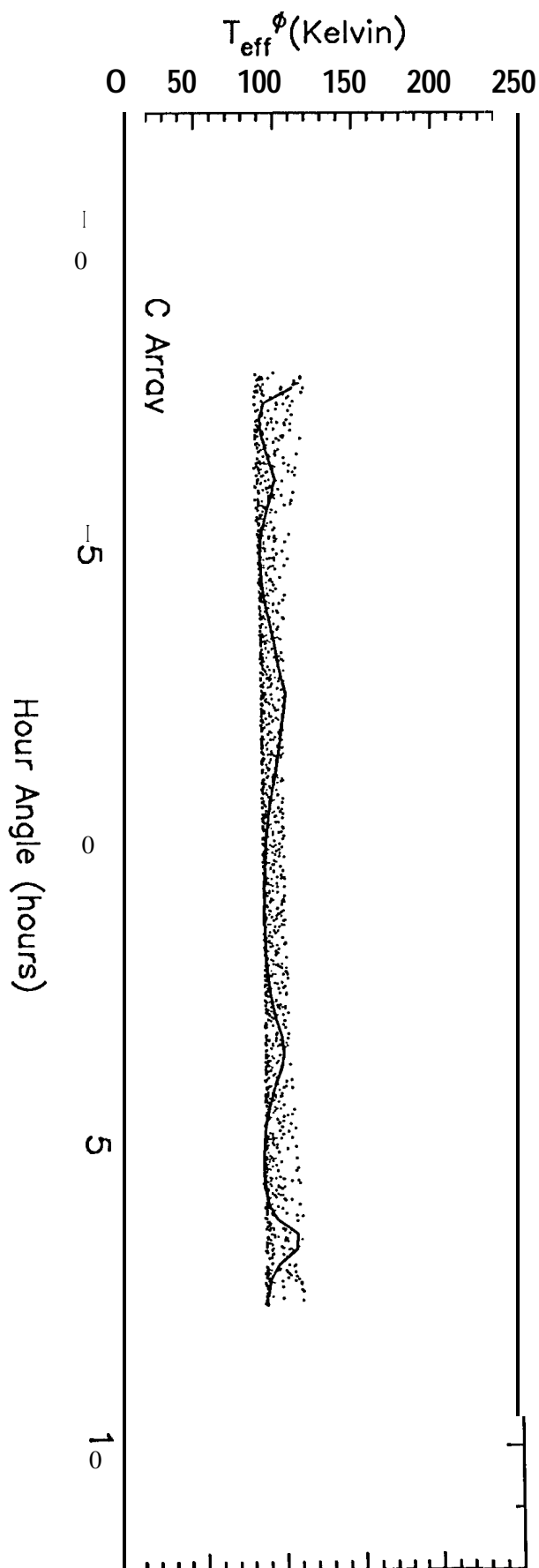
Figure

(a)

T_{eff}^{ϕ} (Kelvin)

0   50   100   150   200   250

A Array

Hour Angle (hours)

(b)

T_{eff}^{ϕ} (Kelvin)

0   50   100   150   200   250

B Array

Hour Angle (hours)

Figure 1

(c)

T_{eff}^{ϕ}(Kelvin)

0   50   100   150   200   250

C Array

−10   −5   0   5   10

Hour Angle (hours)

(d)

T_{eff}^{ϕ}(Kelvin)

0   50   100   150   200   250

D Array

−10   −5   0   5   10

Hour Angle (hours)